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Modal Logic Interpretation of Dempster-Shafer Theory: An Infinite Case

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ABSTRACT

A modal logic interpretation of belief and plausibility measures defined on infinite sets is established. As a special case, a modal logic interpretation of necessity and possibility measures defined on infinite sets is also established. It is proven in both cases that the interpretation is complete. These results establish, in effect, modal logic interpretations of the Dempster-Shafer theory and possibility theory.

KEYWORDS: ***modal logic, Dempster-Shafer theory, possibility theory, belief measures, plausibility measures, necessity measures, possibility measures***

1. INTRODUCTION

The idea of connecting Dempster-Shafer theory and, especially, possibility theory with modal logic (or related provability logic) is by no means new [1, 3–5, 7, 12, 22, 27, 28, 34, 35]. However, previous efforts to develop this idea have concentrated, by and large, on investigating possibilistic logic and the logic of belief functions, and on determining their connections with established systems of modal logic [7, 34]. This was done not only for numerical forms of the theories but also for their qualitative or comparative variants [1, 3, 4, 12, 14]. Our motivation is different. We have tried to establish the usual semantics of modal logic as a possible unifying framework, within which various uncertainty theories can be interpreted and compared. Therefore our focus has been on interpretation of uncertainty theories within modal logic. This research project

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was initiated by Resconi et al. [24] and further developed in a series of papers [15, 20, 25].

All previous work in this area (including our own) is restricted to finite versions of uncertainty theories and finite models of modal logic. Contrary to the previous work, the aim of this paper is to develop a modal logic interpretation of belief and plausibility measures defined on infinite sets (frames of discernment). As is well known, these measures form the basis of evidence theory [32], which is lately referred to as Dempster-Shafer theory. Clearly, we have to use infinite models of modal logic for this purpose. As a special case, a modal logic interpretation is also established for necessity and possibility measures defined on infinite sets. As is well known, these measures form the basis of possibility theory [6].

Our work is closely related to the work of Ruspini [27, 28], who seems to have been the first one to realize that the probability of necessity behaves like a belief function. He utilizes this observation to develop an interpretation of both basic evidence functions of Dempster-Shafer theory and Dempster's rule of combination based on the epistemic logic of Hintikka [16]. In addition to differences in motivation, our work extends Ruspini's contribution in three directions. First, Ruspini utilizes an epistemic logic that is equivalent to the modal system S5, while our interpretation is based on the much weaker system D [17, 18]. Second, Ruspini does not provide a proof of completeness of his interpretation, while we do; that is, we show that for any given belief measure there is a model of modal logic satisfying our requirements that yields the given belief measure. Third, we cover both finite and infinite frames of discernment and models of modal logic, while Ruspini deals only with the finite case. On the other hand, we do not address the interpretation of Dempster's rule of combination.

Ruspini also provides an interpretation of possibility distributions [29–31]. His interpretation is based on graded accessibility relations, called similarity relations. On the contrary, we utilize our interpretation of Dempster-Shafer theory and the well-known relation between necessity/possibility measures and belief/plausibility measures to characterize a class of models of modal logic that yield necessity and possibility measures.

Also connected with our work is the idea of probability of provability. This idea was informally suggested by Pearl [21–23] as a possible interpretation of Dempster-Shafer theory, and further developed by Smets [35]. The connection is based on the fact that certain kinds of provability correspond to necessity in some systems of modal logic [2]. Since neither Pearl nor Smets is sufficiently specific about the kind of provability they deal with, it is hard to characterize this connection more precisely. Contrary to our work, Pearl and Smets deal only with finite domains and do not prove completeness of their interpretation. However, Smets considers the problem of conditioning, which we do not address in this paper.

For the sake of completeness we should also mention the work of Fagin and Halpern [8–11] on logics for reasoning about knowledge and probability. Their setting is somewhat similar to ours, but they are interested in the development of a complete formal system to reason about knowledge and probability, and not in the interpretation of Dempster-Shafer theory.

Although we briefly introduce the concept of a model of modal logic in the next section, basic knowledge of modal logic [17] as well as some background in Dempster-Shafer theory [32, 33] and possibility theory [6] is desirable for full understanding of this paper.

2. BASICS OF MODAL LOGIC

Modal logic is an extension of classical propositional logic. It can be characterized as a logic of logical necessity and possibility. Its language consists of the set of atomic propositions or propositional variables; logical connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$; modal operators of necessity, $\Box$, and possibility, $\Diamond$; and supporting symbols (,), { , }, ... Objects of interest are *formulas*:

- an atomic proposition is a formula;
- if p and q are formulas, then so are $\neg p$, $p \vee q$, $p \wedge q$, $p \rightarrow q$, $p \leftrightarrow q$, $\Box p$, $\Diamond p$.

When developed formally, different modal systems are characterized by different sets of axioms (i.e. special formulas) and inferential rules. Since we are not interested in developing formal systems in this paper, we omit discussion of this matter.

The meaning of a formula is its truth value in a given context. Various contexts are usually expressed in terms of models of modal logic. A *model*, $\mathbf{M}$, of modal logic is a triple

$$\mathbf{M} = \langle W, R, V \rangle,$$

where W, R, V denote, respectively, a set of possible worlds, a binary relation on W, and a value assignment function, by which truth (T) or falsity (F) is assigned to each atomic proposition in each possible world (i.e. $V : W \times Q \longrightarrow \{\mathrm{T}, \mathrm{F}\}$, where Q is the set of all atomic propositions). The value assignment function is inductively extended to all formulas in the usual way, the only interesting cases being

$$V(w, \Box p) = \mathrm{T} \quad \text{iff} \quad \text{for all } u \in W, \quad \langle w, u \rangle \in R \text{ implies } V(u, p) = \mathrm{T}$$

and

$$V(w, \Diamond p) = \mathrm{T} \quad \text{iff} \quad \text{there is } u \in W \text{ such that } \langle w, u \rangle \in R \text{ and } V(u, p) = \mathrm{T}.$$

The relation R is usually called an *accessibility relation*; we say that world u is accessible to world w when $\langle w, u \rangle \in R$. Different systems of modal logic are characterized by different additional requirements on the accessibility relation R. In our further considerations in this paper we assume that R is serial. R is *serial* iff for all $w \in W$ there is $u \in W$ such that $\langle w, u \rangle \in R$. Formally, this requirement corresponds to the system D [17].

3. BASICS OF DEMPSTER-SHAFER THEORY

Let Ω, Δ denote nonempty sets and $\mathcal{P}(\Omega)$, $\mathcal{P}(\Delta)$ their power sets. A class $\mathcal{C}$ of subsets of a set Ω is called a *multiplicative class* if $A \cap B$ is in $\mathcal{C}$ whenever A and B are in $\mathcal{C}$. A class $\mathcal{E}$ of subsets of a set Ω is called an *additive class* if $A \cup B$ is in $\mathcal{E}$ whenever A and B are in $\mathcal{E}$. Suppose $\mathcal{C}$ is a multiplicative subclass of $\mathcal{P}(\Omega)$ containing both $\emptyset$ and Ω, and suppose $\mathcal{D}$ is a multiplicative subclass of $\mathcal{P}(\Delta)$ containing both $\emptyset$ and Δ. We call $h : \mathcal{C} \longrightarrow \mathcal{D}$ a $\cap$-*homomorphism* if $h(\emptyset) = \emptyset$, $h(\Omega) = \Delta$, and $h(A \cap B) = h(A) \cap h(B)$ for all $A, B \in \mathcal{C}$.

We call a function Bel that maps a multiplicative subclass $\mathcal{C}$ of $\mathcal{P}(\Omega)$, which contains both $\emptyset$ and Ω, into the real interval [0, 1] a *belief measure* if the following holds:

1. $\text{Bel}(\emptyset) = 0$,
2. $\text{Bel}(\Omega) = 1$,
3. $\text{Bel}(A) \geq \sum\{(-1)^{|I|+1}\text{Bel}(\bigcap_{i \in I} A_i) \mid \emptyset \neq I \subseteq \{1, \ldots, n\}\}$ for every collection $\{A, A_1, \ldots, A_n\} \subseteq \mathcal{C}$ such that $A_i \subseteq A$ $(i = 1, \ldots, n)$ and for every n.

We call a function Pl that maps a additive subclass $\mathcal{E}$ of $\mathcal{P}(\Omega)$, which contains both $\emptyset$ and Ω, into the real interval [0, 1] a *plausibility measure* if the following holds:

1. $\text{Pl}(\emptyset) = 0$,
2. $\text{Pl}(\Omega) = 1$,
3. $\text{Pl}(A) \leq \sum\{(-1)^{|I|+1}\text{Pl}(\bigcup_{i \in I} A_i) \mid \emptyset \neq I \subseteq \{1, \ldots, n\}\}$ for every collection $\{A, A_1, \ldots, A_n\} \subseteq \mathcal{E}$ such that $A \subseteq A_i$ $(i = 1, \ldots, n)$ and for every n.

REMARK 1 Notice that if Bel is a belief measure on $\mathcal{C}$, then the function Pl defined on the additive class $\mathcal{E} = \{\overline{A} \mid A \in \mathcal{C}\}$ by $\text{Pl}(A) = 1 - \text{Bel}(\overline{A})$ is a plausibility measure. ($\overline{A}$ denotes the complement of A in Ω.)

THEOREM 1 (Shafer [33]) *Let $\mathcal{C}$ be a multiplicative subclass of $\mathcal{P}(\Omega)$ containing both $\emptyset$ and Ω. A function* Bel, *mapping $\mathcal{C}$ into* [0, 1], *is a belief measure if and only if there exists a set Δ, an algebra $\mathcal{D}$ of subsets of Δ, a finitely additive probability measure P on $\mathcal{D}$, and a $\cap$-homomorphism $h : \mathcal{C} \longrightarrow \mathcal{D}$ such that* $\text{Bel} = P \circ h$.

If not specified otherwise in the following, we assume $\mathcal{C}$ to be $\mathcal{P}(\Omega)$.

4. INTERPRETATION OF DEMPSTER-SHAFER THEORY

Let X denote some nonempty universal set (frame of discernment). The set of atomic propositions Q consists of all propositions of the form e_A, where A is an arbitrary subset of X. The proposition e_A is supposed to mean that a given incompletely characterized element of X lies within A. We also assume that each particular world (of some given model) gives its own unique answer to the classification question, i.e., one and only one formula $e_{\{x\}}$ is true in each world, where $x \in X$. This, and proper relations among valuations of e_A for different A, are guaranteed by the following requirements on valuation function V of any given model of modal logic:

- $V(w, e_X) = \mathrm{T}$ for all $w \in W$;
- $V(w, e_A \to e_B) = \mathrm{T}$ for all $w \in W$ and for all $A, B \in \mathcal{P}(X)$ such that $A \subseteq B$;
- $V(w, e_A \to \neg e_C) = \mathrm{T}$ for all $w \in W$ and for all $A, C \in \mathcal{P}(X)$ such that $A \cap C = \emptyset$;
- $V(w, e_{A \cup B} \leftrightarrow (e_A \vee e_B)) = \mathrm{T}$ for all $w \in W$ and for all $A, B \in \mathcal{P}(X)$.

To be able to develop a modal logic interpretation of Dempster-Shafer theory also for the infinite case, we add to the classical models of modal logic a finitely additive probability measure on the set of possible worlds. From now on, a model of modal logic is meant to be the quadruple

$$\mathbf{M} = \langle W, R, V, P \rangle,$$

where W, R, and V are as above, and P is a finitely additive probability measure on W. [We implicitly assume that P is defined on the whole power set $\mathcal{P}(W)$.]

THEOREM 2 *Given a model* $\mathbf{M} = \langle W, R, V, P \rangle$ *of modal logic that satisfies the above stated requirements regarding* R *and* V, *the model yields a belief measure given by*

$$\mathrm{Bel}_M(A) = P(\{w \in W \mid V(w, \Box e_A) = \mathrm{T}\})$$

and a plausibility measure given by

$$\mathrm{Pl}_M(A) = P(\{w \in W \mid V(w, \Diamond e_A) = \mathrm{T}\})$$

for all $A \in \mathcal{P}(X)$. *That is, the belief measure of* A *can be viewed as the probability of worlds for which* $\Box e_A$ *is true, and similarly the plausibility measure of* A *can be viewed as the probability of worlds for which* $\Diamond e_A$ *is true.*

Proof In the standard modal logic, the equivalence $\Diamond p \leftrightarrow \neg\Box\neg p$ is valid for any formula p [17]. From our requirements on valuation function, it also follows that $\neg e_A \leftrightarrow e_{\overline{A}}$ for any $A \in \mathcal{P}(X)$. From these facts and from the additivity of P, we have

$$\begin{aligned}
\mathrm{Pl}_M(A) &= P(\{w \in W \mid V(w, \Diamond e_A) = \mathrm{T}\}) \\
&= P(\{w \in W \mid V(w, \neg\Box\neg e_A) = \mathrm{T}\}) \\
&= P(W) - P(\{w \in W \mid V(w, \Box e_{\overline{A}}) = \mathrm{T}\}) = 1 - \mathrm{Bel}_M(\overline{A})
\end{aligned}$$

for all $A \in \mathcal{P}(X)$. So, by Remark 1, it is sufficient to show that Bel_M is a belief measure. Define a mapping $h : \mathcal{P}(X) \longrightarrow \mathcal{P}(W)$ such that

$$h(A) = \{w \in W \mid V(w, \Box e_A) = \mathrm{T}\}$$

for all $A \in \mathcal{P}(X)$. It is obvious that

$$V(w, e_{A \cap B} \leftrightarrow (e_A \wedge e_B)) = \mathrm{T} \qquad \text{for all } w \in W \text{ and for all } A, B \in \mathcal{P}(X)$$

follows from the four requirements on the valuation function. Utilizing another well-known equivalence from modal logic [17],

$$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q),$$

we get

$$\begin{aligned} h(A \cap B) &= \{w \in W \mid V(w, \Box e_{A \cap B}) = \mathrm{T}\} \\ &= \{w \in W \mid V(w, \Box e_A \wedge \Box e_B) = \mathrm{T}\} \\ &= \{w \in W \mid V(w, \Box e_A) = \mathrm{T}\} \cap \{w \in W \mid V(w, \Box e_B) = \mathrm{T}\} \\ &= h(A) \cap h(B) \end{aligned}$$

for all $A, B \in \mathcal{P}(X)$. From the first requirement on valuation functions we also get $h(X) = W$, and from the first and the third requirements we have $h(\emptyset) = \emptyset$. All together, h is a $\cap$-homomorphism of X into W. By using Theorem 1, we get the conclusion that Bel_M is a belief function. ∎

5. COMPLETENESS

In this section, we show the completeness of our interpretation of Dempster-Shafer theory, and illustrate a possible simplification of the construction in the proof of the following theorem by a simple example.

THEOREM 3 *The interpretation of Dempster-Shafer theory introduced in Theorem 2 is complete, i.e., for every belief measure* Bel *(or plausibility measure* Pl*) on $\mathcal{P}(X)$, there is a model* **M** *of modal logic, satisfying the above requirements, such that*

$$\mathrm{Bel}(A) = \mathrm{Bel}_M(A),$$

for every $A \in \mathcal{P}(X)$.

Proof Using Theorem 1, we know that there is a set Y, a finitely additive probability P on Y, and a $\cap$-homomorphism $h : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ such that $\mathrm{Bel}(A) = P(h(A))$ for all $A \in \mathcal{P}(X)$. Inspecting Shafer's proof of Theorem 1 [33] (based on results of Revuz and Huneycutt [19, 26]), we know we can take $Y = \mathcal{P}(X)$ and $h(A) = \mathcal{P}(A)$ for all $A \in \mathcal{P}(X)$, so that there is a finitely additive probability P on $\mathcal{P}(X)$ such that $\mathrm{Bel}(A) = P(\mathcal{P}(A))$ for all $A \in \mathcal{P}(X)$.

Consider the following model of modal logic:

$$\mathbf{M} = \langle W, R, V, P' \rangle,$$

where

- $W = \mathcal{P}(X) - \{\emptyset\}$;
- R is such that $\langle A, B \rangle \in R$ iff $A, B \in \mathcal{P}(X) - \{\emptyset\}$ and $B \subseteq A$;
- V is such that $V(B, e_A) = \mathrm{T}$ iff $\delta(B) \in A$, for all $B \in W$ and $A \in \mathcal{P}(X)$, where δ is an arbitrary fixed mapping that maps W into X in such a way that $\delta(B) \in B$ for all $B \in W$; and
- P' is restriction of P on W. [It is still a finitely additive probability, since $0 = \mathrm{Bel}(\emptyset) = P(\{\emptyset\})$.]

Clearly, $\mathbf{M}$ satisfies the above requirements regarding R and V. It remains to show that $\mathrm{Bel}_M(A) = \mathrm{Bel}(A)$ for all $A \in \mathcal{P}(X)$. For all $B \in W$, the accessible worlds are all nonempty subsets of B, which means that $\Box e_A$ is true in B if and only if $B \subseteq A$. From this observation and the definitions of Bel_M and P', we have

$$\begin{aligned}\mathrm{Bel}_M(A) &= P'(\{w \in W \mid V(w, \Box e_A) = \mathrm{T}\}) \\ &= P'(\{B \in \mathcal{P}(X) - \{\emptyset\} \mid B \subseteq A\}) \\ &= P(\mathcal{P}(A) - \{\emptyset\}) = P(\mathcal{P}(A)) = \mathrm{Bel}(A)\end{aligned}$$

for all $A \in \mathcal{P}(X)$. ■

The construction introduced in Theorem 3 requires quite a big set of possible worlds, which in a particular situation need not be necessary. To illustrate this point, consider the following simple example. Let the universal set X be the set of all real numbers, let Y be the set $\{\alpha, \beta\}$, and let P be a probability measure on Y induced by the probability distribution $p_\alpha = 0.32$ and $p_\beta = 0.68$. Suppose, we want to find a modal logic model for a belief function Bel given by $\mathrm{Bel}(A) = P(h(A))$ for all $A \in \mathcal{P}(X)$, where h is the $\cap$-homomorphism of $\mathcal{P}(X)$ into $\mathcal{P}(Y)$ given by the formula

$$h(A) = \begin{cases} \emptyset & \text{iff} \quad [3, 7.2] \not\subseteq A \text{ and } [9, 10] \not\subseteq A, \\ \alpha & \text{iff} \quad [3, 7.2] \subseteq A \text{ and } [9, 10] \not\subseteq A, \\ \beta & \text{iff} \quad [3, 7.2] \not\subseteq A \text{ and } [9, 10] \subseteq A, \\ Y & \text{iff} \quad [3, 7.2] \subseteq A \text{ and } [9, 10] \subseteq A \end{cases}$$

for all $A \in \mathcal{P}(X)$. (Bel is a belief function by Theorem 1.)

In this case, we can get a significantly simpler model than by the construction in the proof of the Theorem 3. Define $\mathbf{M} = \langle W, R, V, P' \rangle$, where $W = X \cup Y$,

$$R = \{\langle \alpha, z \rangle \mid z \in [3, 7.2]\} \cup \{\langle \beta, y \rangle \mid y \in [9, 10]\} \cup \{\langle x, x \rangle \mid x \in W\},$$

$$\begin{aligned} V(x, e_A) = \mathrm{T} \qquad \text{iff} \quad & x \in X \text{ and } x \in A \text{ or } x = \alpha \text{ and } 5 \in A \\ & \text{or } x = \beta \text{ and } 9.5 \in A, \end{aligned}$$

for all $A \in \mathcal{P}(X)$, and $P'(A \cup B) = P(B)$ for all $A \in \mathcal{P}(X)$ and all $B \in \mathcal{P}(Y)$.

Let us look, for example, at the interval [2, 9.3]:

$$\text{Bel}([2, 9.3]) = P(h([2, 9.3])) = P(\alpha) = 0.32$$

and

$$\begin{aligned}\text{Bel}_M([2, 9.3]) &= P'(\{w \in W \mid V(w, \Box e_{[2,9.3]}) = \text{T}\}) \\ &= P'(\{\alpha\} \cup [2, 9.3]) = P(\alpha) = 0.32.\end{aligned}$$

In this example the cardinality of W is the continuum, which is definitely smaller than the cardinality of $\mathcal{P}(X) - \{\emptyset\}$.

6. POSSIBILISTIC CASE

In this section, as in [20] for the finite case, we examine interpretations of necessity and possibility measures.

We call a function Nec that maps $\mathcal{P}(X)$ into the real interval [0, 1] a *necessity measure* if and only if

$$\text{Nec}(\emptyset) = 0,$$

$$\text{Nec}(X) = 1,$$

$$\text{Nec}\left(\bigcap_{t \in T} A_t\right) = \inf_{t \in T} \text{Nec}(A_t)$$

for any $\{A_t\}_{t \in T} \subseteq \mathcal{P}(X)$, where T is an arbitrary nonempty index set. A function Pos that maps $\mathcal{P}(X)$ into the real interval [0, 1] is called a *possibility measure* if and only if

$$\text{Pos}(\emptyset) = 0,$$

$$\text{Pos}(X) = 1,$$

$$\text{Pos}\left(\bigcup_{t \in T} A_t\right) = \sup_{t \in T} \text{Pos}(A_t)$$

for any $\{A_t\}_{t \in T} \subseteq \mathcal{P}(X)$, where T is an arbitrary nonempty index set.

REMARK 2 It is easy to show that even in the infinite case every necessity measure is a belief measure and every possibility measure is a plausibility measure and that the well-known relationship

$$\text{Pos}(A) = 1 - \text{Nec}(\overline{A}), \qquad (*)$$

holds for all $A \in \mathcal{P}(X)$ [i.e., for every necessity measure Nec on X, the function Pos given by $(*)$ is a possibility measure, and vice versa].

Due to these facts, we can use our interpretation of Dempster-Shafer theory also for possibility theory. The question is, as in [20] for the finite case, what class of models corresponds to possibility theory. Not surprisingly, the answer is almost the same: the models with transitive and connected accessibility relation and finitely additive probability continuous from above on $\mathcal{W} = \{\{w \in W \mid V(w, \Box e_A) = \text{T}\} \mid A \in \mathcal{P}(X)\}$. [$R$ is transitive iff $\langle u, v\rangle \in R$ and $\langle v, w\rangle \in R$ imply $\langle u, w\rangle \in R$ for all $u, v, w \in W$; it is connected iff for all $u, v \in W$, $\langle u, v\rangle \in R$ or $\langle v, u\rangle \in R$. A set function $P : \mathcal{P}(X) \longrightarrow [0, 1]$ is continuous from above on $\mathcal{A} \subseteq \mathcal{P}(X)$ iff $\{E_n\} \subseteq \mathcal{A}$, $E_1 \supseteq E_2 \supseteq \cdots$ and $\bigcap_{n=1}^{\infty} E_n \in \mathcal{A}$ imply $\lim_n P(E_n) = P(\bigcap_{n=1}^{\infty} E_n)$.] We justify this claim by the following two theorems.

THEOREM 4 *Given a model* $\mathbf{M} = \langle W, R, V, P\rangle$ *of modal logic such that* V *satisfies the above requirements,* R *is transitive and connected, and* P *is continuous from above on* $\mathcal{W}$, *the model yields a necessity measure given by*

$$\text{Nec}_M(A) = P(\{w \in W \mid V(w, \Box e_A) = \text{T}\})$$

and a possibility measure given by

$$\text{Pl}_M(A) = P(\{w \in W \mid V(w, \Diamond e_A) = \text{T}\})$$

for all $A \in \mathcal{P}(X)$.

Proof Due to Remark 2, we only have to show that

$$\text{Nec}_M\left(\bigcap_{t\in T} A_t\right) = \inf_{t\in T} \text{Nec}_M(A_t)$$

for any $\{A_t\}_{t\in T} \subseteq \mathcal{P}(X)$, where T is an arbitrary nonempty index set.

For $C \in \mathcal{P}(X)$, the set $\{w \in W \mid V(w, \Box e_C) = \text{T}\}$ is denoted by $\mathcal{T}_C$. Observe that, for any $A, B \in \mathcal{P}(X)$, it is true that $\mathcal{T}_A \subseteq \mathcal{T}_B$ or $\mathcal{T}_B \subseteq \mathcal{T}_A$. Suppose this is not true. Then, there are $u \in \mathcal{T}_A$ and $v \in \mathcal{T}_B$ such that $u \notin \mathcal{T}_B$ and $v \notin \mathcal{T}_A$. However, by connectedness we have $\langle u, v\rangle \in R$ or $\langle v, u\rangle \in R$. If $\langle u, v\rangle \in R$, then from transitivity of R and the fact that $v \notin \mathcal{T}_A$ we get $u \notin \mathcal{T}_A$, which is contradiction. Similarly, from $\langle v, u\rangle \in R$ we would get $v \notin \mathcal{T}_B$. Therefore, $\mathcal{T}_A \subseteq \mathcal{T}_B$ or $\mathcal{T}_B \subseteq \mathcal{T}_A$.

This implies that, given an arbitrary fixed family $\{A_t\}_{t\in T} \subseteq \mathcal{P}(X)$ for some nonempty index set T, the family $\{\mathcal{T}_{A_t}\}_{t\in T}$ can be ordered by set inclusion. Moreover, we can find a sequence $\{\mathcal{T}_{A_i}\}_{i=1}^{\infty}$ such that $\mathcal{T}_{A_i} \in \{\mathcal{T}_{A_t}\}_{t\in T}$, $\mathcal{T}_{A_i} \supseteq \mathcal{T}_{A_{i+1}}$ for all i, and $\bigcap_{i=1}^{\infty} \mathcal{T}_{A_i} = \mathcal{T}_{\bigcap_{t\in T} A_t}$. Therefore, we have

$$\text{Nec}_M\left(\bigcap_{t\in T} A_t\right) = P\left(\mathcal{T}_{\bigcap_{t\in T} A_t}\right) = P\left(\bigcap_{i=1}^{\infty} \mathcal{T}_{A_i}\right)$$

and also (by continuity from above of P on $\mathcal{W}$)

$$P\left(\bigcap_{i=1}^{\infty} \mathcal{T}_{A_i}\right) = \lim_i P\left(\mathcal{T}_{A_i}\right) = \inf_{t\in T} P\left(\mathcal{T}_{A_t}\right) = \inf_{t\in T} \text{Nec}_M(A_t). \qquad \blacksquare$$

THEOREM 5 *The interpretation of possibility theory introduced in Theorem 4 is complete, i.e., for any necessity measure* Nec *on* X, *there is a model*

$\mathbf{M} = \langle W, R, V, P\rangle$ of modal logic such that R is connected and transitive, V satisfies our requirements, P is continuous from above on $\mathcal{W}$, and

$$\mathrm{Nec}(A) = \mathrm{Nec}_M(A)$$

for all $A \in \mathcal{P}(X)$.

Proof Assume that we are given some necessity measure Nec on X. Define the dual possibility measure Pos to Nec in the usual way:

$$\mathrm{Pos}(A) = 1 - \mathrm{Nec}(\overline{A})$$

for all $A \in \mathcal{P}(X)$.

Consider the model $\mathbf{M} = \langle W, R, V, P\rangle$, where $W = X$,

$$\langle u, v\rangle \in R \quad \text{iff} \quad \mathrm{Pos}(\{u\}) \leq \mathrm{Pos}(\{v\}),$$

$$V(u, e_A) = \mathrm{T} \quad \text{iff} \quad u \in A$$

for all $u \in X$ and $A \in \mathcal{P}(X)$, and $P(\mathcal{T}_A) = \mathrm{Nec}(A)$. P can be arbitrarily extended to $\mathcal{P}(X)$; the extension is not relevant. Next we show that P is well defined. Assume that for some $A, B \in \mathcal{P}(X)$ we have $\mathcal{T}_A = \mathcal{T}_B$ and $\mathrm{Nec}(A) \neq \mathrm{Nec}(B)$. Without loss of generality, we can assume that $\mathrm{Nec}(A) < \mathrm{Nec}(B)$. This means that there is some $x \in X$ such that $x \in \overline{A}$, $x \in B$, and $\mathrm{Pos}(\{x\}) > \mathrm{Pos}(\overline{B})$. This implies that for any $y \in X$ such that $\mathrm{Pos}(\{y\}) \geq \mathrm{Pos}(\{x\})$ it has to be true that $y \in B$. However, this means that $x \in \mathcal{T}_B$ and $x \notin \mathcal{T}_A$, which is a contradiction. So P is well defined. It is clear from our definitions that V satisfies our requirements and R is transitive and connected. Continuity of P from above on $\mathcal{W}$ follows from the definition of P and from the property $\mathrm{Nec}(\bigcap_{t\in T} A_t) = \inf_{t\in T} \mathrm{Nec}(A_t)$ of the necessity measure Nec. Finally, by definition of P, we have

$$\mathrm{Nec}(A) = P(\mathcal{T}_A) = P(\{w \in W \mid V(w, \Box e_A) = \mathrm{T}\}) = \mathrm{Nec}_M(A)$$

for all $A \in \mathcal{P}(X)$. ■

7. CONCLUSIONS

Modal logic interpretations of Dempster-Shafer theory and possibility theory were established for the finite case in our previous papers [15] and [20], respectively; this paper extends these interpretations to the infinite case. Although our intuitions about the two theories are not substantially enhanced by these extensions, we believe that the presented results are significant for at least two reasons. First, they show that it is indeed possible to extend the proposed interpretations to the infinite case and at the same time keep all the basic properties. This is not obvious, since there are examples of notions for which the generalization to the infinite case is not so direct (e.g. the Shannon entropy). Secondly, these results furnish further support for suitability of the proposed interpretations.

We intend to employ these interpretations, for both finite and infinite frames of discernment, to some of the foundational issues in the two theories that are still not fully resolved. We also intend to investigate various practical utilizations of the interpretations.

Results presented in this paper undoubtedly add further credibility to the research program of developing a hierarchical uncertainty metatheory based upon modal logic, which is stated in [24]. However, many more results will be needed before the stated aims are achieved.

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